Vector Integration:

How to integrate vectors $\rightarrow$ Reduce vector integral to scalar integral.

(i) Line Integrals – Three type of line integral.

$$\int_C \phi \, d\vec{r} \quad \text{— (1)}$$

$$\int_C \vec{V}\cdot d\vec{r} \quad \text{— (2)} \qquad ; \; d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

length element

$$\int_C \vec{V}\times d\vec{r} \quad \text{— (3)}$$

Integral is over Contour C

Contour C $\rightarrow$ May be open or closed (loop)

$\rightarrow$ ϕ is scalar, $\vec{V}$ is vector.

$\rightarrow$ The second integral is most important of the three

$\rightarrow$ ~~To evaluate the~~ consider the second integral

$$\int_C \vec{V}\cdot d\vec{r}$$

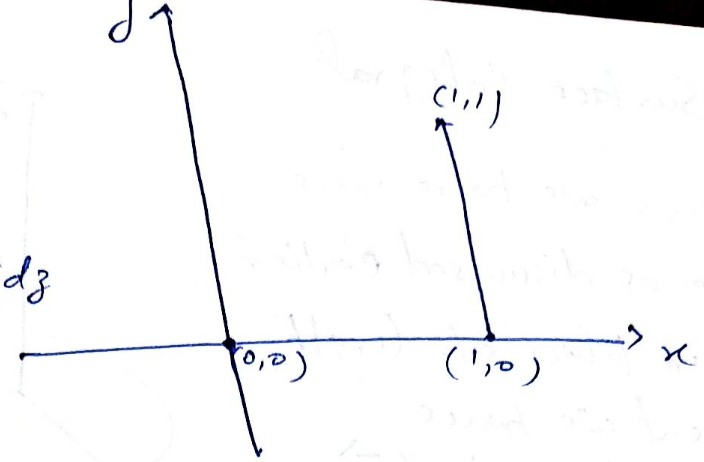
If we want to calculate work done by a force $\vec{F}$ which varies along the path:

$$W = \int \vec{F}\cdot d\vec{r}$$

Consider the force applied on a body is $\vec{F} = -y^2\hat{x} + x\hat{y}$

We want to calculate work done from $(0,0)$ to $(1,1)$.

$$W = \int_{0,0}^{1,1} \vec{F} \cdot d\vec{r}$$



$$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

$$\vec{F} \cdot d\vec{r} = (-y^2\hat{x} + xy\hat{y}) \cdot (\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz)$$

$$= -y^2 dx + x\,dy$$

$$W = \int_{0,0}^{1,1} (-y^2 dx + x\,dy)$$

limit for x is ranging from $(0,1)$ and for y it is same $(0,1)$.

So we separate both the integrals corresponding to x & y.

$$W = -\int_0^1 y^2 dx + \int_0^1 x\,dy$$

To evaluate first integral we need to know what is y for $x \to (0,1)$. See Fig. for x going from 0 to 1 y is 0

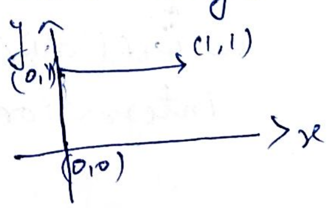
$$W = -\int_0^1 0\,dx + \int_0^1 x\,dy = \int_0^1 x\,dy$$

Now we need to calculate what is x for y going from 0 to 1.

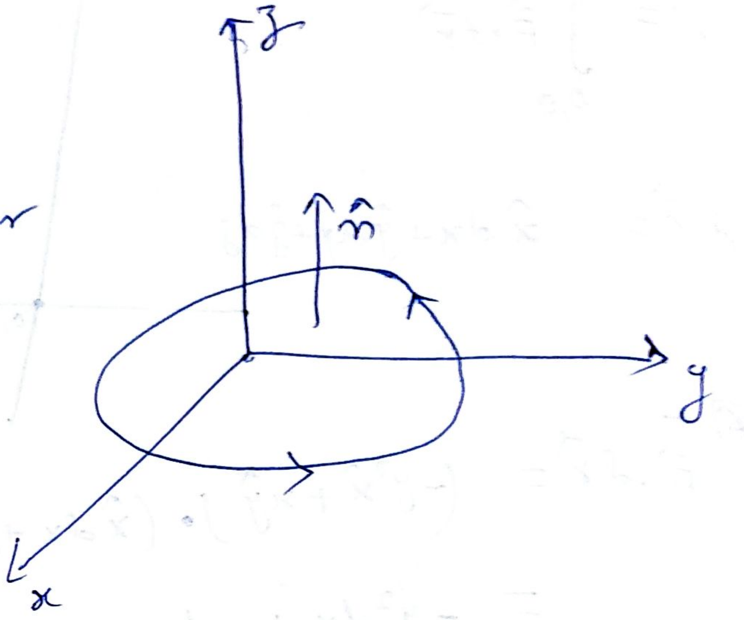
In this case x is 1.

$$\boxed{W = \int_0^1 1\,dy = 1.}$$

Note: changing the path. Work done W will change.
So W is path dependent.
See figure. End points are same but path is changed. Hence W will change.



(ii) Surface Integrals.



Here we have same form as discussed earlier but in place of length element we have area element $d\vec{\sigma}$

$d\vec{\sigma} \rightarrow$ vector

$d\vec{\sigma} = \hat{n}\, dA$, $\hat{n}$ - unit normal, for Fig above it is indicating positive normal.

The three types are

$$\int_S \phi\, d\vec{\sigma}$$

$$\int_S \vec{V}\cdot d\vec{\sigma} \longrightarrow \text{most encountered.}$$

$$\int_S \vec{V}\times d\vec{\sigma}$$

$\int_S \vec{V}\cdot d\vec{\sigma}$ — denotes flow or flux through the given surface

(iii) Volume Integrals:

For volume element $d\tau$ we can write volume ~~element~~ integral.

$$\int_V \vec{V}\, d\tau = \hat{x}\int_V V_x\, d\tau + \hat{y}\int_V V_y\, d\tau + \hat{z}\int_V V_z\, d\tau$$

$d\tau = dx\,dy\,dz \rightarrow$ scalar quantity

We will discuss more examples and elaborate the vector integration in next note.